Dumbbell Librations in Elliptic Orbits

HANS B SCHECHTER*
The Rand Corporation, Santa Monica, Calif

This paper discusses and analyzes the first-order effect of orbital eccentricity on the planar tumbling or oscillatory motion of a dumbbell-shaped satellite This has been done by assuming that the angular orientation angle Ψ can be represented by a power series in eccentricity *e*, in which the coefficient of the e^0 term was set equal to the circular solution Ψ available from earlier investigations The differential equation for the coefficient of the e^1 term is shown to be of an inhomogeneous Mathieu type, the particular solutions of which can be readily obtained if certain weak restrictions are placed on the initial magnitude of the dumbbell's angular velocity The analysis indicated that the orientation of the satellite in the elliptic orbit can differ substantially from the one determined for the circular orbit

THE effect of gravitational gradient torques on the attitude behavior of dumbbell-shaped satellites has been studied in a number of papers [1–8] In the earlier studies, the angular excursions of the dumbbell's axis were held down to small values, but this restriction was later lifted to permit inclusion of rotational as well as oscillatory motion In most cases, a circular orbit was assumed The influence of orbital eccentricity e seems to have been assessed only for oscillations of small amplitude It is of interest to see in which way orbital eccentricity modified some of the general results obtained in the coplanar circular case The mathematical treatment in the present approach differs from that of some of the previous papers that considered the same basic problem by the fact that the small angle approximation has been replaced by an expansion around a finite angular rotation

In the notation of Fig 1, the equations of motion of the dumbbell are

$$\ddot{r} - r\dot{\theta}^2 = (-\mu/2)[(r - l\cos\Psi)r_1^{-2} + (r + l\cos\Psi)r_2^{-2}] \quad (1)$$

$$2m(d/dt)[r^2\dot{\theta} + l^2(\dot{\theta} + \dot{\Psi})] = 0 \quad (2)$$

$$(d/dt)[l^2(\dot{\theta} + \dot{\Psi})] = (\mu r l \sin\Psi)/2[r_2^{-3} - r_1^{-3}] \quad (3)$$

Equation (2) indicates that the combined angular momentum of the motion is conserved

If we now take e, $l/r \ll 1$, and ignore the small effect of satellite dumbbellness on the elliptic orbital path of its center of mass,[6] it can be shown[9] that the substitution into Eq (3) of an expansion of the form

$$\Psi = \Psi_c + \sum_{n=1}^{\infty} \Psi_n e^n$$

and collection of like coefficients of e leads to the following set of equations:

$$\ddot{\Psi}_c + \tfrac{3}{2}n^2 \sin 2\Psi_c = 0 \quad (4)$$

$$\ddot{\Psi}_c + e\ddot{\Psi}_1 + \tfrac{3}{2}n^2[1 + 3e\cos M] \times [\sin 2(\Psi + e\Psi_1)] = 2en^2 \sin M \quad (5)$$

where $M = nt$ = mean anomaly

Approximating $\sin 2(\Psi + e\Psi_1)$ by $(\sin 2\Psi + 2e\Psi_1 \cos 2\Psi)$, using Eq (4), and switching to the mean anomaly M as the independent variable gives

$$\Psi'' + \tfrac{3}{2}\sin 2\Psi = 0 \quad (6)$$

$$\Psi_1'' + (3\cos 2\Psi)\Psi_1 = 2\sin M - \tfrac{9}{2}\cos M \sin 2\Psi_c \quad (7)$$

where primes denote differentiation with respect to the mean anomaly M

For the case $\Psi_c(0)$, Schindler and Moran[5,6] have shown that the form of the solution of Eq (6) depends on the initial value of the angular velocity Ψ_i' Two distinct regions can be distinguished, depending on whether the motion is one of oscillation or of tumbling For oscillatory motion $|\Psi_{ci}'| < 3^{1/2}$, and the solution is represented by $\sin\Psi_c = k\,sn\, 3^{1/2}M$ The modulus k of the elliptic function is equal to $\Psi_i'/3^{1/2}$ For the case of tumbling motion, which corresponds to the condition $|\Psi_i'| > 3^{1/2}$, the solution is given by $\sin\Psi = sn\, \Psi_i'M$ and $k = 3^{1/2}/\Psi_i'$ The degenerate case $\Psi_i' = \pm 3^{1/2}$ represents a motion of infinite period in which the dumbbell approaches asymptotically the horizontal position $\Psi = \pm\pi/2$

A plot of this solution, taken from Ref 6, is reproduced for convenience in Fig 2

Effect of Eccentricity on Tumbling Motion

The solution of Eq (7) is facilitated if the range of values of Ψ_{ci}' is broken up into two separate regions For large values of Ψ_{ci}', say $\Psi_i' \geq 3$, Fig 2 shows that the oscillatory ripple around the constant-slope line of Ψ vs M has nearly disappeared, so that it is reasonable to approximate $\sin\Psi_c$ by $\sin\alpha M$, where M is the mean anomaly, and α is the average value of the angular velocity $d\Psi_c/dM$, which is assumed to be larger than or equal to 3 For convenience of analysis, α will be considered an integer Letting $s = \alpha M$, Eq (7) becomes

$$(d^2/ds^2)\Psi_1 + [(3/\alpha^2)\cos 2s]\Psi_1 = (2/\alpha^2)\sin s/\alpha - (9/2\alpha^2)\cos(s/\alpha)\sin 2s \quad (8)$$

which is an inhomogeneous Mathieu equation of the form

$$(d^2/dx^2)z + [a - 2q\cos 2x]z = f(x) \quad (9)$$

with $a = 0$ and $q = -3/(2\alpha^2)$ In view of the assumption $\alpha \geq 3$, all solutions to Eq (8) are stable and oscillatory in nature and consist of Mathieu functions of fractional order [10] The complete solution to Eq (8) is

$$\Psi_1 = C_1(s)\Psi_{11}(s) + C_2(s)\Psi_{12}(s) + \Psi_1(0)\bar{\Psi}_{11}(s) + \Psi_1'(0)\bar{\Psi}_{12}(s) \quad (10)$$

In the notation of Ref 10, with β denoting the order of the

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* Research Engineer Member AIAA

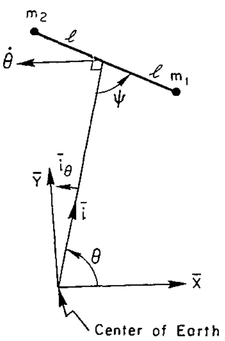

Fig 1 Geometry of motion

Mathieu function, Ψ_{11} and Ψ_{12} are the nonperiodic bounded solutions of the homogeneous portion of Eq (8):

$$\Psi_{11} = Ce_\beta(s,-q) = \sum_{j=-\infty}^{\infty} (-1)^j A_{2j}{}^\beta \cos(2j+\beta)s \tag{11}$$

$$\Psi_{12} = Se_\beta(s,-q) = \sum_{j=-\infty}^{\infty} (-1)^j A_{2j}{}^\beta \sin(2j+\beta)s$$

whereas $\bar{\Psi}_{11}$ and $\bar{\Psi}_{12}$ are a set of fundamental solutions chosen such that $\bar{\Psi}_{11}(0) = (d/ds)\bar{\Psi}_{12}(0) = 1$ and $(d/ds)\bar{\Psi}_{11}(0) = \bar{\Psi}_{12}(0) = 0$

Also,

$$C_1(s) = \int_0^s h(\xi)\Psi_{12}(\xi)W^{-1}(\xi)d\xi$$

$$C_2(s) = \int_0^s h(\xi)\Psi_{11}(\xi)W^{-1}(\xi)d\xi$$

$$h(\xi) = (2/\alpha^2)\sin\xi/\alpha - (9/2\alpha^2)\cos\xi/\alpha \sin 2\xi$$

$$W(\xi) = W[\Psi_{11}(\xi),\Psi_{12}(\xi)] = \text{Wronskian of } \Psi_{11} \text{ and } \Psi_{12}$$

Since all of the initial conditions on the motion are assumed to have been satisfied by the circular-orbit solution Ψ, we have $\Psi_1(0) = \Psi_1'(0) = 0$, and the complementary portion of the solution consequently vanishes The Wronskian W will be a constant because Eq (8) has no first derivative of Ψ_1 As a numerical example, we can consider the case when $\alpha = 3$, and hence $q = -3/2\alpha^2 = -0\ 1667$ When all of the operations are carried out, we finally end up with the solution

$$\Psi_1 = 1\ 00028 - 1\ 0565 \sin 0\ 3333s + 0\ 10892 \sin 1\ 6667s - 0\ 01276 \sin 2\ 3333s - 0\ 00124 \sin 4\ 3333s - 0\ 00112 \sin 3\ 6667s + \tag{12}$$

The result is plotted in Fig 3, which shows the behavior of Ψ, Ψ_1, and Ψ_1' as a function of M for one orbital period of the dumbbell

The magnitude of the perturbation caused by an orbital eccentricity of $e = 0\ 1$ is shown by the dashed curve It is apparent that orientation errors as large as 80° could occur during the first orbital period

Effect of Eccentricity on Oscillatory Motion

As Ψ_i' is reduced from a value of about 3 and approaches $3^{1/2}$, the accuracy of the approximation $\Psi \approx \Psi_{ci}' M$ becomes progressively worse, and the foregoing solution ceases to describe the motion Neglecting the narrow region around the condition $\Psi_{ci}' = 3^{1/2}$, we consider now the form of the solution of Eq (8) for $\Psi_i' < 3^{1/2}$ The functions $\cos 2\Psi$ and $\sin 2\Psi$ appearing in the differential equation (7) can be written in the form[11]

$$\cos 2\Psi = 1 - 2\sin^2\Psi = 1 - 2k^2 sn^2 3^{1/2}M$$
$$\sin 2\Psi = 2\sin\Psi \cos\Psi = -2k[d/d(3^{1/2}M)]cn 3^{1/2}M \tag{13}$$

and the elliptic functions sn and cn expanded in Fourier series:

$$\frac{kK}{2\pi} sn 3^{1/2}M = \sigma^{1/2}(1-\sigma)^{-1}\sin 3^{1/2}\frac{\pi M}{2K} + \sigma^{3/2}(1-\sigma^3)^{-1}\sin 3(3)^{1/2}\frac{\pi M}{2K} + \sigma^{5/2}(1-\sigma^5)^{-1}\sin 5(3)^{1/2}\frac{\pi M}{2K} + \ldots = \sum_{j=0}^{\infty} \sigma^{j+1/2}(1-\sigma^{2j+1})^{-1}\sin(2j+1)3^{1/2}\frac{\pi M}{2K} \tag{14a}$$

$$\frac{kK}{2\pi} cn 3^{1/2}M = \sum_{j=0}^{\infty} \sigma^{j+1/2}(1+\sigma^{2j+1})^{-1}\cos(2j+1)3^{1/2}\frac{\pi M}{2K} \tag{14b}$$

$$\sigma = \exp[-\pi K'/K]$$

$$K = K(k) = \int_0^{\pi/2} (1 - k^2\sin^2\phi)^{-1/2}d\phi$$

$$K' = K(k') \qquad k'^2 = 1 - k^2$$

For sufficiently small values of σ, the retention of the leading term in each of Eqs (14a) and (14b) might give sufficient accuracy A rough idea of the size of σ is obtained by taking, for instance, the case $\Psi_i' = 1\ 66$ as a rough indication of an upper bound on the angular velocity This corresponds to equality in orbital and librational periods For this case, $k = 1\ 66/3^{1/2} = 0\ 958$, $k' = 0\ 2868$, $K(k) = 2\ 8$, $K(k') \cong 1\ 6$, and $\sigma = \exp[-1\ 6\pi/2\ 8] = 0\ 165$

As Ψ_i' decreases, k will decrease and k' will increase, so that K will be reduced and K' increased; this will lead to a decrease in the size of σ Thus, for values of $\Psi_i' < 1\ 66$ we find that $\sigma < 0\ 165$

Placing an upper limit of 1 66 on Ψ_{ci}', it appears that the retention of only one term in the Fourier expansions would be satisfactory in the present case Under this approximation, terms that are of magnitude σ have been neglected in comparison with unity

A graphical representation of the foregoing approximation is presented in Fig 4, which shows the functions snu and cnu, together with the first expansion term of their respective Fourier series, for the case $\Psi_i' = 1\ 66$ This corresponds to the extreme case for which the approximate representation is least accurate As Ψ_i' approaches zero, the approximation continues to improve, until at $\Psi_i' = 0$ the first Fourier term identically represents the elliptic functions

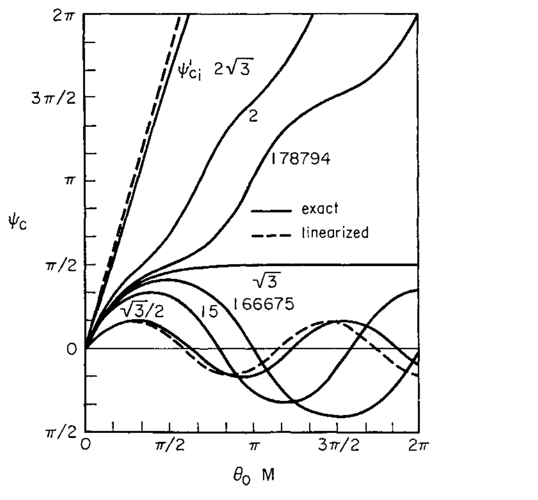

Fig 2 Time history of rotational motion in a circular orbit [6]

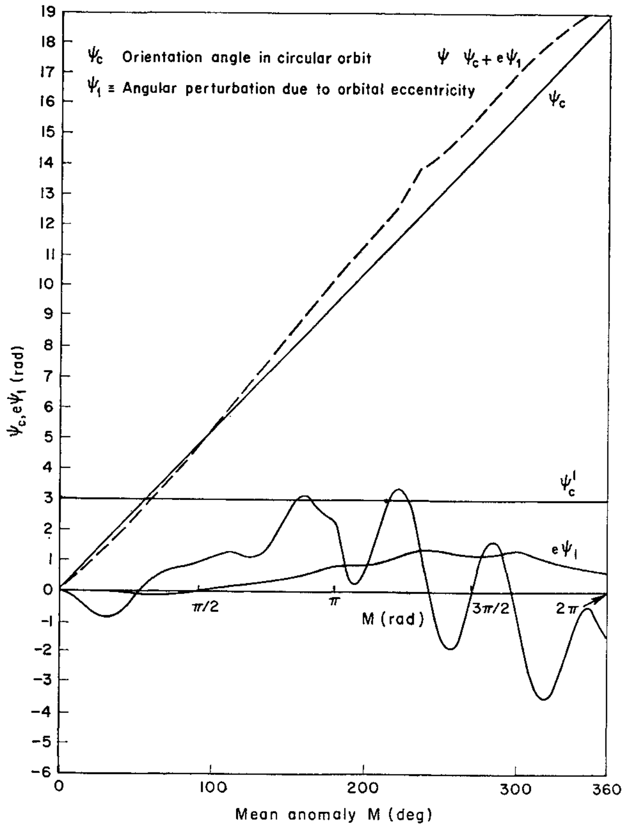

Fig 3 Solution for the tumbling case with $\alpha = 3$, $e = 0$ 1

Introduction of the independent variable $z = 3^{1/2}\pi M/2K$ converts Eq (7) to the form

$$(d^2/dz^2)\Psi_1 + [a - 2q\cos 2z]\Psi_1 =$$
$$(8K^2/3\pi^2)\sin(2K/3^{1/2}\pi)z - 6\sigma^{1/2}[(1+\sigma)]^{-1} \times$$
$$[\sin(2K/3^{1/2}\pi + 1)z + \sin(1 - 2K/3^{1/2}\pi)z] \quad (15)$$

where $a = 4K^2/\pi^2 - 16\sigma(1-\sigma)^{-2}$ and $q = -8\sigma(1-\sigma)^{-2}$

As in the last case, here too the motion of libration is governed by an inhomogeneous Mathieu equation The bounded oscillatory form of the forcing function suggests immediately that for finite values of Ψ_{ci}' the nature of the forced solution of Eq (15) depends on the stability or instability exhibited by the solution to the homogeneous Mathieu equation This question is easily resolved with the aid of Fig 5, which presents in the plane of a vs q the stable and unstable regions of the solution to Mathieu's equation Because of symmetry with respect to the a axis, the right side of the Mathieu plane has been used The curves Ce_0, Se_1, and Ce_1 denote, respectively, the loci of zeroth- and first-order periodic solutions to Mathieu's equations and represent the transition curves between the stable and unstable regions shown The directed dashed curve indicates the trace of the (a,q) point of Eq (15) as the initial value of angular velocity Ψ_{ci}' tends toward zero Except for a narrow region around $a = 1$, $q = 0$, the homogeneous solution is essentially oscillatory and divergent as z increases, making the complete solution unstable Although the scale of Fig 5 does not make this apparent, a simplified analysis shows that the dashed curve approaches the point $a = 1$, $q = 0$ with a slope $da/d|q| = -2$ Since the Se_1 curve exhibits at this point a slope of -1, it appears that the dashed curve penetrates into the stable region of the plane as Ψ_{ci}' assumes vanishingly small values

In the limit, as $\Psi_{ci}' \to 0$, which corresponds to the case of an initially nonrotating dumbbell moving along an elliptic orbit, we note that $K \to \pi/2$, $a \to 1$, $K' \to \infty$, and $\sigma \to 0$

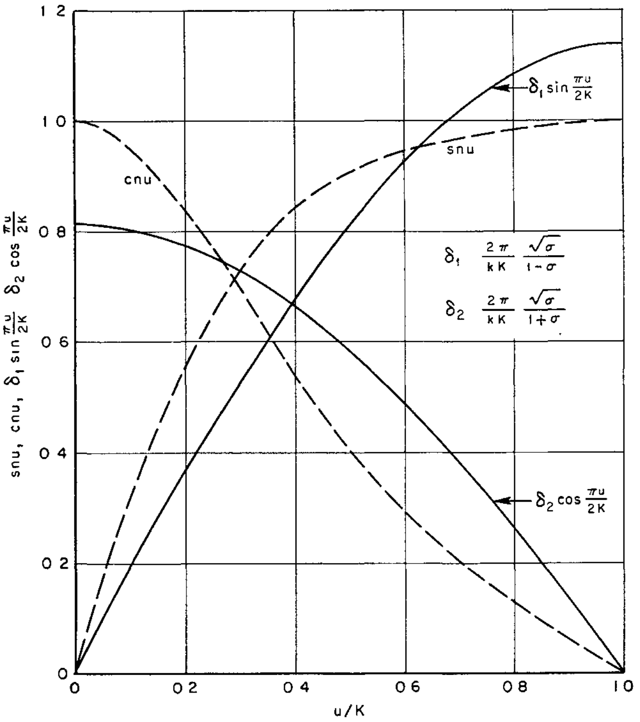

Fig 4 The elliptic functions *sn* and *cn* and their first Fourier expansion terms

Equation (15) is then reduced to the form

$$(d^2/dz^2)\Psi_1 + \Psi_1 = \tfrac{2}{3}\sin(1/3^{1/2})z \quad (16)$$

The complementary solution of Eq (16) will be bounded and periodic, but the forced solution, although bounded too, will not exhibit any periodicity The form of the arguments of the trigonometric terms in the forcing function precludes the occurrence of resonance conditions

Conclusion

The following concluding remarks can be made which summarize the previous results

It has been shown that the correction terms due to eccentricity, which must be superimposed on the angular-position time behavior of the same satellite when moving along a

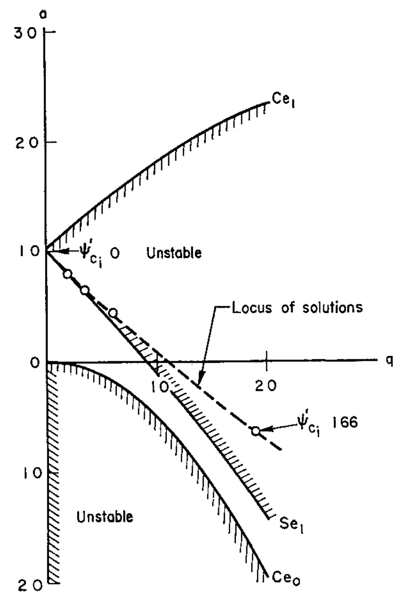

Fig 5 Effect of initial angular velocity ψ_{ci}' on the location of the solution point in the a,q plane

circular orbit, are always bounded and oscillatory for all values of initial angular velocity in excess of roughly three times the orbital angular velocity Depending on the magnitude of the eccentricity, these perturbations could be quite appreciable

When the initial motion is one of pendulous oscillations, the first-order effect of eccentricity is to introduce a divergent oscillatory term into the time behavior of the first perturbation term and thus to cause the orientation angle in the elliptic orbit to differ significantly from that assumed in the circular orbit Because of possible phase difference in the time behavior of the circular orientation angle Ψ_c and the perturbed correction term Ψ_1, it is still possible for the complete solution not to exhibit an actual boundless increase in amplitude as long as the restrictions imposed by linearity are not violated

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